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ORIGINAL PAPER

The structure character of essentially disconnected polyomino graphs

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Abstract This paper deals with the structure of essentially disconnected polyomino graphs. It is proved that for an essentially disconnected polyomino graph, the normal components induced digraph is acyclic and connected. The lower bond for the number of normal components of an essentially disconnected polyomino graph is investigated. Moreover, the essentially disconnected polyomino graphs with two or three normal components are classified and constructed.

Keywords Construction · Polyomino graph · Matching · Fixed bond · Essentially disconnected · Normal

1 Introduction

Polyomino graphs [1], which are also called chessboards [2] or square-cell configurations [3], have attracted some mathematicians' considerable attention, because many interesting combinatorial subjects can be produced from them, such as hypergraphs [1], domination problem [2,4], rook polyominal [5], etc. In addition, Motoyama and Hosoya obtained some interesting results by introducing king and domino polyomials, which can be applied in statistical physics and in modeling problems of surface chemistry [5,6].

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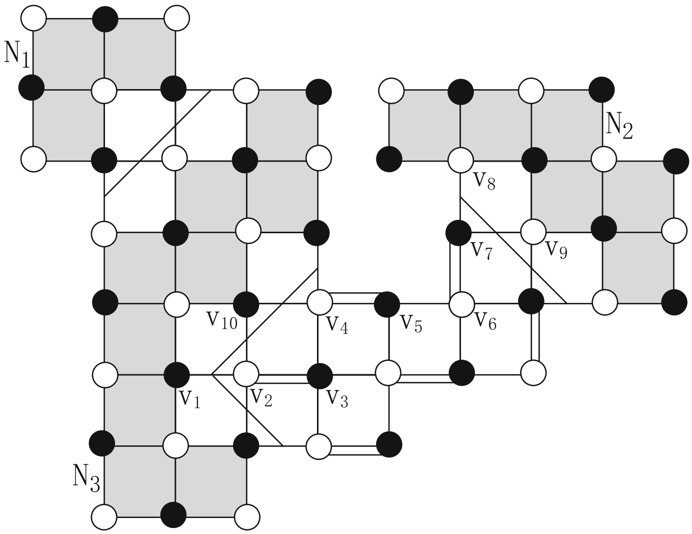


Fig. 1 An essentially disconnected polyomino graph G

A *Polyomino graph* is bipartite and 2-colorable. In the following, we make the convention that all the vertices of a polyomino graph G in question have been colored black and white so that the end vertices of any edge have different colors, and denote G by $G = (W, B)$, where W and B are the sets of white vertices and black vertices, respectively. A *perfect matching* of a graph G is an independent edge set such that each vertex of G is incident with one of the edges. All polyomino graphs mentioned later have perfect matchings unless otherwise stated. An edge of G is called a *fixed single* (*fixed double*) *bond* if it belongs to no (each) perfect matching of G. A *fixed bond* is either a fixed single bond or a fixed double bond. A polyomino graph without (with) fixed bonds is said to be *normal* (*essentially disconnected*). Let G be a polyomino graph with fixed bonds. The *unfixed subgraph* of G is the subgraph induced by the non-fixed bonds of G. Each connected component of the unfixed subgraph is normal and is called a *normal component*. Similarly, we define *fixed subgraph* of G to be the subgraph induced by the fixed bonds of G and each connected component of which is called a *fixed component* (cf. Fig. 1, the normal components are shadowed, while the fixed double bonds are shown by double lines.)

Clearly, the unfixed subgraph of an essentially disconnected polyomino graph can be obtained by deleting of the fixed double bonds together with their end vertices and deleting of the fixed single bonds without their end vertices. It is found that an essentially disconnected polyomino graph has at least two connected components and each of them is a normal polyomino graph [7]. They also found that if an essentially disconnected polyomino graph has an unit square as one of its normal components, then it has at least three normal components [7]. No more results about the structure of essentially disconnected polyomino graphs have been known. In this paper, we investigate the structure characters of essentially disconnected polyomino graphs by means of investigating its normal components induced digraph (formal definition is given later), and the relationships among normal components and try to classify and construct them.

2 Structure of essentially disconnected polyomino graphs

Let G be a graph. A *boundary* vertex (edge) of G is a (an) vertex (edge) that lies on the boundary of G. Let H a subgraph of G. $G - H$ denotes the subgraph of G obtained by deleting the vertices of H and their incident edges.

Definition 2.1 Let v a vertex of an essentially disconnected polyomino graph G. Vertex v is called a spreading vertex if it lies on the boundary of a normal component N of G, and is adjacent to a vertex of $G - N$.

Definition 2.2 Let N be a normal component of an essentially disconnected polyomino graph $G = (W, B)$. N is said to be of type $B(W)$, if all the spreading vertices of N are black (white); N is said to be of type M, if N contains both white spreading vertices and black spreading vertices; N is said to be of type I, if none of the vertices of N is a boundary vertex of G.

Note that a normal component of type I is certainly a normal component of type M. But a a normal component of type M need not be a normal component of type I.

As Fig. 1 shows, N_1, N_2 and N_3 are of type B, W and M, respectively.

Definition 2.3 Denoted by $\mathcal{G}_n$ the set of essentially disconnected polyomino graphs with exactly n normal components. For $G \in \mathcal{G}_n$, denote by $n_W(G), n_B(G), n_M(G)$ the number of normal components of type W, type B and type M, respectively.

Note that a normal component of type I is certainly a normal component of type M. For any $G \in \mathcal{G}_n$, we have $n_W(G) + n_B(G) + n_M(G) = n$.

Definition 2.4 Let $G = (W, B)$ be an essentially disconnected polyomino graph. A path P of G is said to be an F-path of G if it satisfies : 1. the edges of P are alternately fixed single bonds and fixed double bonds; 2. the end vertices of P are spreading vertices.

Definition 2.5 Let G be an essentially disconnected polyomino graph, N_1 and N_2 be two normal components of G. N_1 is said to be *incident* with an F-path P if one of the end vertices of P belongs to N_1 . We say N_1 connects to N_2 if N_1 and N_2 are incident with the same F-path. We say N_1 properly connects to N_2 if N_1 and N_2 are incident with the same F-path P; and $N_1 \cap P$ is a black vertex, while $N_2 \cap P$ is a white vertex.

In Fig. 1, $v_9v_7v_6v_5v_4v_{10}$ and $v_1v_2v_3v_4v_5v_6v_7v_8$ are F-paths, while $v_1v_2v_3v_4v_5v_6 v_7v_8$ is a strict F-path; N_3 connects to N_1, while N_3 properly connects to N_2.

Let $G = (W, B) \in \mathcal{G}_n$, $N_1, N_2, \ldots, N_n$ be the normal components of G. We construct the normal components induced digraph ($NC-$induced digraph) of G, denoted by $I_N(G)$, as follows (cf. Fig. 2):

1. the vertex set of $I_N(G)$ is $\{N_1, N_2, \ldots, N_n\}$;
2. for any two vertices $N_i, N_j \in \{N_1, N_2, \ldots, N_n\}$, there is an arc from N_i to N_j (N_i is said to be the tail of the arc and N_j is said to be the head of the arc) if and only if N_i is properly connected to N_j.

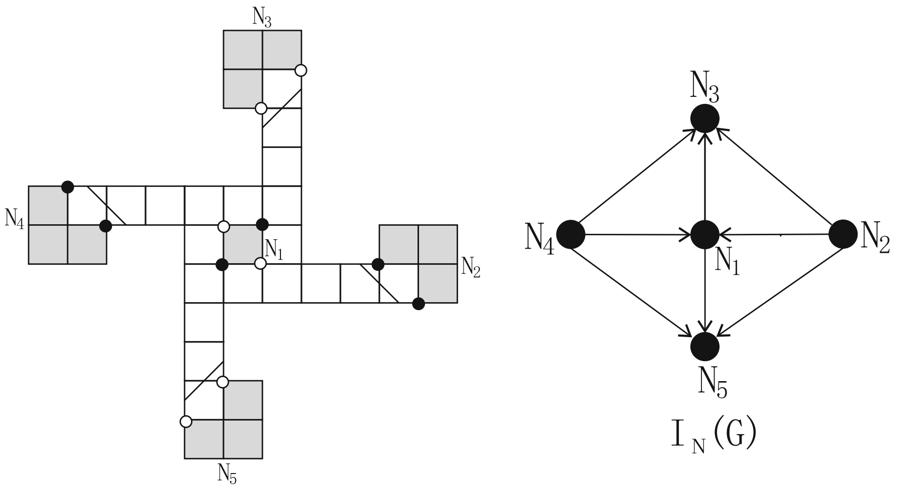


Fig. 2 A polyomino graph G and its NC-induced digraph $I_N(G)$

The indegree $d^-_{I_N(G)}(N)$ of a vertex N in $I_N(G)$ is the number of arcs with head N; the outdegree $d^+_{I_N(G)}(N)$ of N in $I_N(G)$ is the number of arcs with tail N. The following theorem is obvious:

Theorem 2.6 *Let N be a normal component of an essentially disconnected polyomino graph G. Then we have:*

1. $d^+_{I_N(G)}(N) = 0$ *if and only if N is of type W;*
2. $d^-_{I_N(G)}(N) = 0$ *if and only if N is of type B;*
3. $d^+_{I_N(G)}(N) > 0$ *and* $d^-_{I_N(G)}(N) > 0$ *if and only if N is of type M.*

Lemma 2.7 *Let G be an essentially disconnected polyomino graph, and v be a vertex of G. Then v is a spreading vertex if and only if v is incident with at least two non-fixed bonds and at least one fixed single bond.*

Proof Suppose that vertex v is a spreading vertex. By the definition of spreading vertex, v belongs to a normal component N of G. Thus v is incident with at least two edges in N which are non-fixed bonds since N is normal. Note that any edge between N and $G - N$ is a fixed single bond. Hence v is incident with at least one fixed single bond.

Conversely, suppose that v is incident with at least two non-fixed bonds and at least a fixed single bond. Evidently, the two non-fixed bonds are adjacent and belong to a normal component N of G. Moreover, the other end vertex of the fixed single bond is in $G - N$. By the definition of spreading vertex, v is a spreading vertex. Therefore, the lemma holds. □

Lemma 2.8 (see [8]) *A polyomino graph G is normal if and only if G possesses a perfect matching M such that the boundary of G is an M-alternating cycle.*

Theorem 2.9 *For any* $G \in \mathcal{G}_n (n \geq 2)$, $I_N(G)$ *is acyclic.*

Proof By contradiction. Assume that there is a directed cycle C in $I_N(G)$. Suppose that $C = N_{i_1}N_{i_2}\dots N_{i_k}N_{i_1}$, where $i_1, i_2, \dots, i_k$ are pairwise different. For any $1 \le s \le k$, by lemma 2.2.2, there is a perfect matching M_{i_s} of N_{i_s} such that the boundary of N_{i_s} is an M_{i_s}-alternating cycle since N_{i_s} is a normal polyomino graph. Let $M = \sum_{s=1}^{k} M_{i_s} \cup M^*$, where M^* is the set of fixed double bonds of G. No doubt, M is a perfect matching of G such that for each $1 \le s \le k$, the boundary of N_{i_s} is an M-alternating cycle. Note that $i_1, i_2, \dots, i_k$ are pairwise different. One can check that there is an M-alternating cycle containing some fixed bonds, a contradiction. The contradiction is caused by our assumption that there is a directed cycle in $I_N(G)$. So the assumption is false. Therefore, the theorem holds. □

Corollary 2.10 *For any $G \in \mathcal{G}_n (n \ge 2)$, we have: $n_B(G) \ge 1, n_W(G) \ge 1$.*

Proof By Theorem 2.9, $I_N(G)$ is acyclic. So there are two vertices $N_i, N_j \in I_N(G)$ such that the indegree $d^-_{I_N(G)}(N_i) = 0$ and the outdegree $d^+_{I_N(G)}(N_j) = 0$. By Theorem 2.6, the normal component N_i is of type B, and the normal component N_j is of type W. Therefore, $n_B(G) \ge 1, n_W(G) \ge 1$. □

Corollary 2.11 *Let $G = (W, B)$ be an essentially disconnected polyomino graph. If G has a normal component of type M, then G has at least three normal components.*

Proof By Corollary 2.10, $n_B(G) \ge 1$ and $n_W(G) \ge 1$. If $n_M(G) \ge 1$, then the number of normal components of G equals $n_W(G) + n_B(G) + n_M(G) \ge 3$. □

The known theorem due to Wei and Ke [7] is a direct consequence of the above corollary:

Theorem 2.12 [7] *If an essentially disconnected polyomino graph G with more than one unit square has a normal component which is a unit square, then G has at least three normal components.*

Proof Let s be a unit square which is a normal component of G. Since G is a polyomino graph and s is a subgraph of G, there is another unit square s' adjacent to s. Since s is a normal component of G, the two vertices belonging to both s and s' are spreading vertices. Note that these two vertices are adjacent to each other. Thus they are of different colors. Therefore, the normal component s is of type M. By Corollary 2.11, G has at least three normal components. □

Theorem 2.13 *For each positive integer triple (a, b, c) subject to*

$$\begin{cases} a \ge 1 \\ b \ge 1 \\ a + b + c = n \ge 2 \end{cases}$$

there is an essentially disconnected polyomino graph $G \in \mathcal{G}_n (n \ge 2)$ such that $(n_W(G), n_B(G), n_M(G)) = (a, b, c)$.

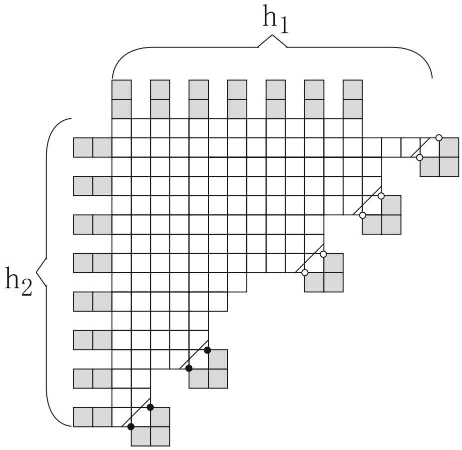


Fig. 3 An illustration for the proof of Theorem 2.13

Proof For each given positive integer triple (a, b, c) subject to

$$\begin{cases} a \geq 1 \\ b \geq 1 \\ a + b + c = n \geq 2 \end{cases}$$

we can construct an essentially disconnected polyomino graph $G \in \mathcal{G}_n (n \geq 2)$ as depicted in Fig. 3, where h_1 and h_2 denote the lengths of the horizontal line and the vertical line, respectively, and satisfy:

$$\begin{cases} h_1 \geq 3a + 3b \\ h_2 \geq 3a + 3b \\ h_1 + h_2 \geq 2c - 1 \end{cases}$$

One can check that $(n_W(G), n_B(G), n_M(G)) = (a, b, c)$. In Fig. 3, $h_1 = 17$, $h_2 = 16$, $a = 3$, $b = 2$, $c = 15$. □

Lemma 2.14 *Let G be an essentially disconnected polyomino graph. Then each fixed bond of G belongs to an F-path.*

Proof Direct each fixed single (double) bond of G from its black (white) end vertex to its white (black) end vertex. It is not difficult to see that by our direction way, each fixed bond e is contained in a maximum directed path P such that the end vertices of P are spreading vertices (otherwise we can expand P at its end vertices). And the edges of P are alternately fixed single bonds and fixed double bonds. Therefore, by the definition of F-path, the lemma holds. □

Lemma 2.15 *Let P_1, P_2 be two F-paths of an essentially disconnected polyomino graph G. If there is a fixed single bond e such that one of its end vertices is on P_1 and the other one is on P_2, then the normal components incident with P_1 or P_2 are connected in $I_N(G)$.*

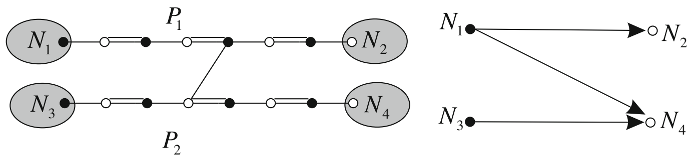


Fig. 4 Illustrations for the proof of Lemma 2.15

Proof Let $N_1, N_2(N_3, N_4)$ be the two normal components incident with $P_1(P_2)$. Clearly, $N_1(N_3)$ is connected to $N_2(N_4)$. Since e is a fixed single bond, e can not be incident with end vertices of P_1 or P_2. In fact, the end vertices of e are incident with a fixed double bond of P_1 and a fixed double bond of P_2, respectively. One can check that in the subgraph $P_1 \cup P_2 \cup \{e\}$, there is an F-path P_3 different from P_1, P_2 such that P_3 connects one of N_1, N_2 and one of N_3, N_4 (cf. Fig. 4). Therefore, N_1, N_2, N_3, N_4 are connected and the lemma holds. □

Theorem 2.16 *Let G be an essentially disconnected polyomino graph. Then $I_N(G)$ is connected.*

Proof Note that the normal components of G are connected by fixed components of G. It suffices to prove that for each fixed component F of G, the normal components $N_F^1, N_F^2, \ldots, N_F^k$ incident with F are connected. For any $1 \leq t \leq k$, there is a fixed single bond e_t of F incident with a spreading vertex of N_F^t. By lemma 2.14, e_t belongs to an F-path P_t. Evidently, one of the components incident with P_t is N_F^t. Thus each normal component incident with F is connected to another one by an F-path which is a subgraph of F. Note that all the F-paths in the fixed component F are connected by fixed single bonds of F. By Lemma 2.15, the theorem holds. □

For convenience, in the following, a polyomino graph G in question is always drawn on the plan so that two edges of each square are horizontal.

Lemma 2.17 *Let G be an essentially disconnected polyomino graph, e be a fixed of G. Then e can not be contained in an M-alternating cycle for any perfect matching M of G.*

Proof Let M be a perfect matching of G and e be a fixed single (double) bond. Suppose that e is contained in an M-alternating cycle C. One can check that $(M \cup C) - (M \cap C)$ is another perfect matching of G in which e is an $M-$double (single) bond, contradicting that e is a fixed single (double) bond. Therefore, e can not be contained in an M-alternating cycle. □

Lemma 2.18 *Let H be a normal component of type I of an essentially disconnected polyomino graph G, e be an edge on the boundary of H. If edge $f \in G - H$ and e belong to the same square s and are parallel, then f is a fixed single bond of G.*

Proof Since H is normal, there is a perfect matching M_H of H such that the boundary of H is an M_H-alternating cycle. Without loss of generality, we may assume that

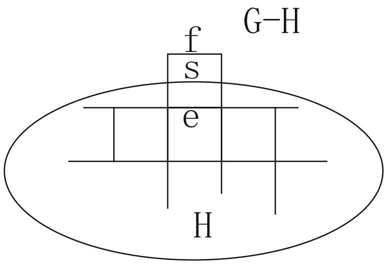


Fig. 5 An illustration for the proof of Lemma 2.18

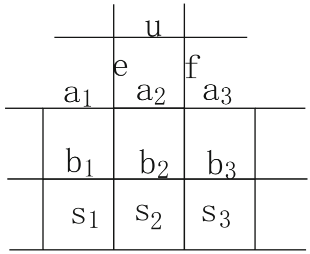


Fig. 6 An illustration for the proof of Lemma 2.19

$e \in M_H$. If f is not a fixed single bond of $G - H$, then there is a perfect matching M' of $G - H$ such that $f \in M'$. Evidently, $M = M_H \cup M'$ is a perfect matching of G. Then s is an M-alternating cycle. Bear in mind that the edges of s except e and f are fixed single bonds of G. Thus we find two fixed single bonds of G contained in an M-alternating cycle of G, contradicting Lemma 2.17. Therefore, f is a fixed single bond of $G - H$. Note that a fixed single bond of $G - H$ is evidently a a fixed single bond of G (Fig. 5). □

Lemma 2.19 *Let H be a normal component of type I of an essentially disconnected polyomino graph G. Then for each of the four possible positions, the top level of H contains at most two squares.*

Proof By contradiction. Assume that for some possible position, the top level of H possesses at least three squares, say s_1, s_2 and s_3 (see Fig. 6). Since H is normal, there is a perfect matching M_H of H such that the boundary $C(H)$ of H is an M_H-alternating cycle. Without loss of generality, we may assume that $b_2 \in M_H$. Let M' be a perfect matching of $G - H$. Then $M = M_H \cup M'$ is a perfect matching of G such that $b_2 \in M$ and the boundary of H is an M-alternating cycle. By Lemma 2.18, a_1, a_2 and a_3 are fixed single bonds. Note that the edges between H and $G - H$ are certainly fixed single bonds of G. Therefore, both e and f are fixed double bonds. One can check that edges e, u, f and a_2 form an alternating cycle, contradicting Lemma 2.17. Consequently, for each of the four possible positions, the top level of H contains at most two squares. □

Lemma 2.20 *Let H be a normal component of type I of an essentially disconnected polyomino graph G. Then for each of the four possible positions, the top level of H contains exactly one square.*

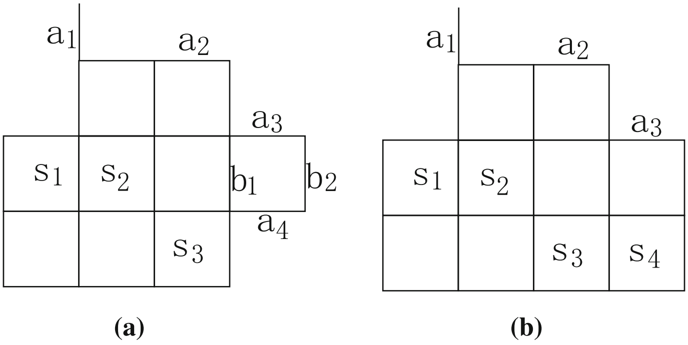


Fig. 7 Illustrations for the proof of Lemma 2.20

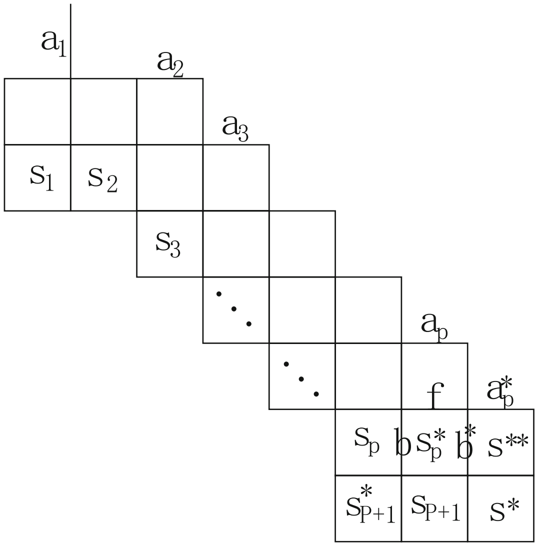
Proof By Lemma 2.19, we may assume that there are two squares s_1 and s_2 on the top level of H for some of the four possible positions. By Lemma 2.18, it is not difficult to see that edges a_1 and a_2 are fixed double bonds of G. If square s_3 does not belong to H, then edges a_3 and a_4 are fixed double bonds (see Fig. 7a). Now we find an alternating cycle consisting of a_3, b_2, a_4 and b_1, contradicting Lemma 2.17. Hence s_3 belongs to H. Suppose s_4 belongs to H too (see Fig. 7b). By a similar reasoning as above, a_3 is a fixed double bond of G, contradicting Lemma 2.18. Therefore, s_4 does not belong to H. Continue this discussion, we find a maximum natural number p in the sense that there is a series of squares lying on the boundary of $H : s_1, s_2, \ldots, s_p$ (see Fig. 6) such that one of the two cases appears: 1. $s^*_{p+1} \in H$ and $s^* \in H$; 2. $s_{p+1} \notin H$.

Suppose that $s^*_{p+1} \in H$ and $s^* \in H$. By a similar reasoning as above, we find a series of fixed double bonds of $G : a_1, a_2, \ldots, a_p$ as depicted in Fig. 8. It is easy to see that $s^*_p \notin H$ (otherwise, a_p is a fixed single bond by Lemma 2.18, a contradiction). If $s^{**} \notin H$, then a^*_p is a fixed double bond, again contradicting Lemma 2.18. Hence $s^{**} \in H$. Note that H is normal, there is a perfect matching M_H of H such that the boundary of H is an M_H-alternating cycle satisfying $b \in M_H$ and $b^* \in M_H$. Now a fixed single bond f is contained in an M_H-alternating cycle consisting of b, g, b^* and f, contradicting Lemma 2.17. Thus, case 1 is impossible. Therefore, $s_{p+1} \notin H$. Similarly, $s^*_p \notin H$ and $s^*_{p+1} \notin H$. We call s_p a corner square of H. Now by turning anti-clockwise for 90 degree, s_p is the top level of G. Continue this discussion as above. We find another corner square of H. Again continue this discussion as above. Eventually, s_1 and s_2 become simultaneously the corner squares of H, contradicting that at each corner of H there is exactly one square. Consequently, for each of the four possible positions, the top level of H contains exactly one square. □

For a polyomino graph G, put a point in the center of each square lying on the boundary of G, and two points are connected by a straight line if and only if the corresponding squares are incident. Thus a polygram is obtained. We call this polygon a boundary polygon.

Theorem 2.21 *Let H be a normal component of type I of an essentially disconnected polyomino graph G. Then the boundary polygon of H is a rectangle.*

Fig. 8 Another illustration for the proof of Lemma 2.20



Proof By Lemma 2.20, for each of the four possible positions, the top level of H contains exactly one square. Let s_0 be the unique square on the top of H for some of the four possible positions. By Lemma 2.18, a is a fixed single bond. Note that b and d are certainly fixed single bonds. Let M be a perfect matching of G. We distinguish two cases:

Case 1 $g \in M$ (see Fig. 9). By a similar reasoning as above, there are a series of squares on the boundary of H : $s_1, s_2, \ldots, s_{n_1-1}$; $s_{n_1}, s_{n_1+1}, \ldots, s_{n_1+n_2-2}$; $s_{n_1+n_2-1}$, $s_{n_1+n_2}, \ldots, s_{n_1+n_2+n_3-3}$; $s_{n_1+n_2+n_3-2}, s_{n_1+n_2+n_3-1}, \ldots, s_{n_1+n_2+n_3+n_4-4} = s_0$, where s_{n_1-1}, $s_{n_1+n_2-2}$, $s_{n_1+n_2+n_3-3}$ and $s_{n_1+n_2+n_3+n_4-4} = s_0$ are corner squares of H. Note that along the boundary of G clockwise, the squares on the boundary of H go down and right from s_0 to s_{n_1-1}; go down and left from s_{n_1-1} to $s_{n_1+n_2-2}$; go up and left from $s_{n_1+n_2-2}$ to $s_{n_1+n_2+n_3-3}$; go up and right from $s_{n_1+n_2+n_3-3}$ to $s_{n_1+n_2+n_3+n_4-4} = s_0$; each time one unit down or up, and one unit right or left. Thus from s_0 to s_{n_1-1}, altogether n_1 units down and n_1 units right. Similarly, from s_{n_1-1} to $s_{n_1+n_2-2}$, altogether n_2 units down and n_2 units left; from $s_{n_1+n_2-2}$ to $s_{n_1+n_2+n_3-3}$, altogether n_3 units up and n_3 units left; from $s_{n_1+n_2+n_3-3}$ to $s_{n_1+n_2+n_3+n_4-4}$, altogether n_4 units up and n_4 units right. Since $s_{n_1+n_2+n_3+n_4-4} = s_0$,we have: $n_1 + n_2 = n_3 + n_4$ (1) (down and up direction)and $n_1 + n_4 = n_2 + n_3$ (2) (right and left direction). By (1)–(2), we have $n_2 = n_4$. Thus $n_1 = n_3$. Therefore, the boundary polygon of H is a rectangle.

Case 2 $f \in M$. It is not difficult to see that $t \notin M$ (otherwise, fixed single bond a is contained in an M-alternating cycle, contradicting Lemma 2.17). Hence $c \in M$. There is a perfect matching of G such that $c \in M$. Notice the symmetry between c and g, it can be dealt with in an entirely similar way as in case 1, and we come to the same conclusion. □

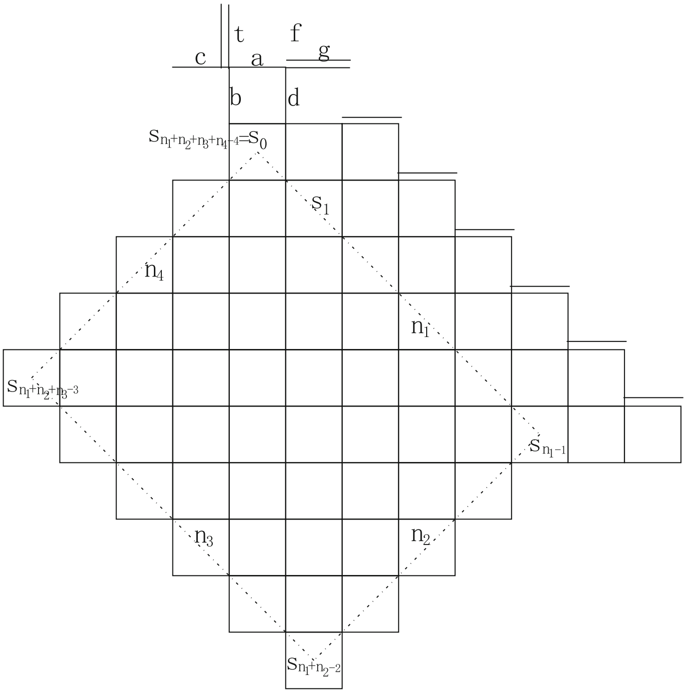


Fig. 9 An illustration for the proof of Theorem 2.21

Let N and N^* be two normal components of a polyomino graph $G = (W, B)$, P and P^* be two F-paths connecting N and N^*, $P \cap N = u$, $P^* \cap N = u^*$, $P \cap N^* = v$ and $P^* \cap N^* = v^*$. Evidently, u and u^* are spreading vertices of N; while v and v^* are spreading vertices of N^*. The closed region bounded by N, P, N^* and P^* is a subgraph of G the boundary of which consists of F-paths P and P^*; the section of boundary of N between u and u^* and the section of boundary of N^* between v and v^*.

Lemma 2.22 *Let N and N^* be two normal components of a polyomino graph $G = (W, B)$, P and $P*$ be two F-paths connecting N and N^*, $P \cap N = u$, $P^* \cap N = u^*$. Let v be a spreading vertex of N in the closed region bounded by N, P, N^* and P^*. Then v, u and u^* have the same color.*

Proof By contradiction. Since $I_N(G)$ is acyclic, u and u^* has the same color (Otherwise, in $I_N(G)$ there would be an arc from N to N^* and an arc from N^* to N, which make a directed cycle of $I_N(G)$). Assume that v has different color with u and u^*. Without loss of generality, let u, u^* be black and v white. By Lemmas 2.7 and 2.14, v belongs to an F-path which connects N to another normal component N_1. As argued above, $N_1 \neq N^*$. One can check that N_1 must be inside the closed region bounded by N, N^*, P and P^*, say R, otherwise there would be a direct loop on vertex N of $I_N(G)$, which contradicts Theorem 2.9 (cf. Fig. 10).

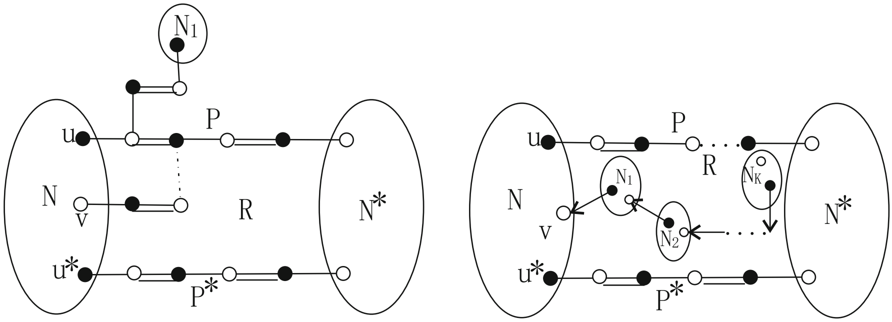


Fig. 10 Illustrations for the proof of Lemma 2.22

Note that normal component N_1 in R is certainly of type I. Hence there is another normal component N_2 in R such that N_2 properly connects to N_1. Again N_2 is of type I. Continue this discussion, we can find in R a series of normal components $N_1, N_2, \ldots, N_k, \ldots$ satisfying that N_{i+1} properly connects to N_i, $i = 1, 2, \ldots$. Since R is finite, the number of different normal components in R is finite. Therefore, we can find two integers s and t such that $1 \le s < t$ and $N_s = N_t$. One can check that there is a cycle in $I_N(G) : N_s = N_t \to N_{t-1} \to \cdots \to N_{s+1} \to N_s$, a contradiction. This contradiction is caused by our assumption that v has different color with u and u'. Therefore, the assumption is false and the lemma holds. □

By Lemma 2.21, for a normal component N of type I, the spreading vertices along the boundary of N can be denoted by $w_1, w_2, \ldots, w_p, b_1, b_2, \ldots, b_q, w'_1, w'_2, \ldots, w'_p, b'_1, b'_2, \ldots, b'_q$ where w_i and $w'_i i = 1, 2, \ldots, p$ are white, b_j and $b'_j j = 1, 2, \ldots, q$ are black.

Lemma 2.23 *Let N be a normal component of type I of a polyomino graph $G = (W, B)$, $w_1, w_2, \ldots, w_p, b_1, b_2, \ldots, b_q, w'_1, w'_2, \ldots, w'_p, b'_1, b'_2, \ldots, b'_q$ are spreadings vertices of N along the boundary of N, where w_i and $w'_i (i = 1, 2, \ldots, p)$ are white, b_j and b'_j $(j = 1, 2, \ldots, q)$ are black. Let $N_{wi}(N_{w'i}), N_{bj}(N_{b'j})$ be normal components connecting N by F-paths with end vertices w_i (w'_i) and b_j $\left(b'_j\right)$, respectively ($i = 1, 2, \ldots, p$; $j = 1, 2, \ldots, q$). Then $N_{w_1}, N_{b_1}, N_{w'_1}$ and $N_{b'_1}$ are pairwise different.*

Proof It is immediate from Lemma 2.22. □

Corollary 2.24 *Both the in-degree and out-degree of each normal components of type I of a polyomino graph $G = (W, B)$ are at least 2 in $I_N(G)$.*

Theorem 2.25 *Let G be a polyomino graph. If G has a normal component of type I, then G has at least five normal components.*

The essentially disconnected polyomino graph depicted in Fig. 11 has five normal components among which one is of type I.

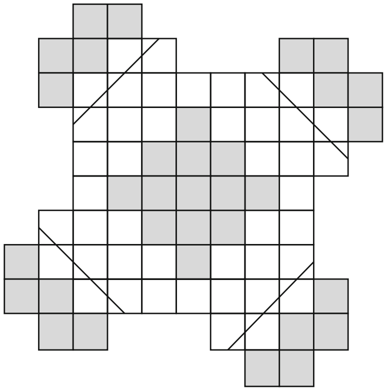


Fig. 11 An essentially disconnected polyomino graph with five normal components

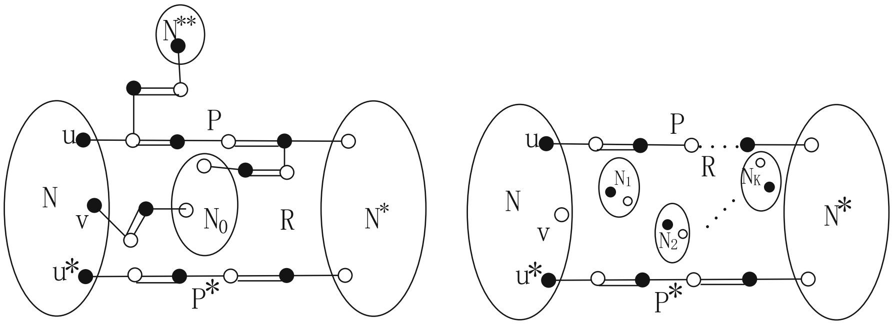


Fig. 12 Illustrations for the proof of Theorem 2.26

Theorem 2.26 *Let N, N^* be two normal components of a polyomino graph $G = (W, B)$. P and P^* are two strict F-paths connecting N and N^*. Then there is no normal component in the region bounded by N, P^*, P and N^*.*

Proof By contradiction. Let R be the region bounded by N, P, N^*, P^*. Assume that there exists a normal component, say N_0, in R. Clearly, N_0 is of type I. By Lemma 2.21, N_0 possesses at least four spreading vertices w_1, b_1, w_1' and b_1' clockwise on its boundary such that w_1, w_1' are white and b_1, b_1' are black. One can check that if a white spreading vertex of N_0 is the end vertex of an F-path which properly connects a normal component N^{**} out of R to N_0, then this F-path must intersect P or P^* and makes another F-path by which N properly connects N_0. Hence, at most one of w_1, w_2, w_3 can spread to an F-path for N_0 to connect N or another normal component out of R. Otherwise, there would be two strict F-paths P_1, P_2 connecting N and N_0 such that (without loss of generality) $w_i \in P_i$, $i = 1, 2$ (cf. Fig. 12).

Note that b_1 is a black spreading vertex of N_0, contradicting Lemma 2.22. Thus, there is at least one white spreading vertex of N_0 that can not spread to an F-path to connect N, N^* or a normal component out of R. So N_0 has at least one white spreading vertex which spreads to an F-path to connect another normal component in R. Let $N_1, N_2, \ldots, N_k$ be the normal components in R, H be the subgraph of $I_N(G)$ induced by $N_1, N_2, \ldots, N_k$, i.e., $H = \langle N_1, N_2, \ldots, N_k \rangle_{I_N(G)}$. By the above result, each vertex

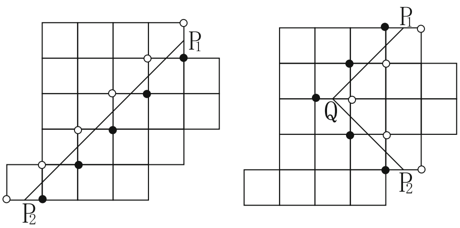


Fig. 13 Illustrations for Definitions 3.27 and 3.28

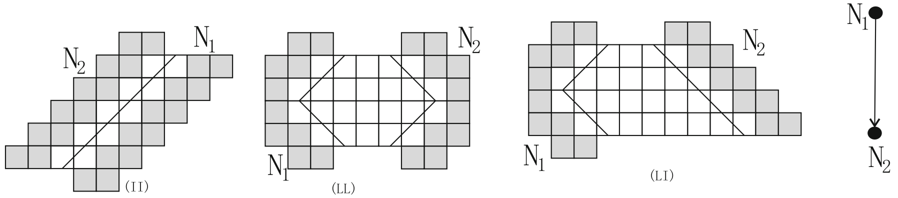


Fig. 14 Illustrations for polyomino graphs of $\mathcal{G}_2$ and their NC-induced digraph

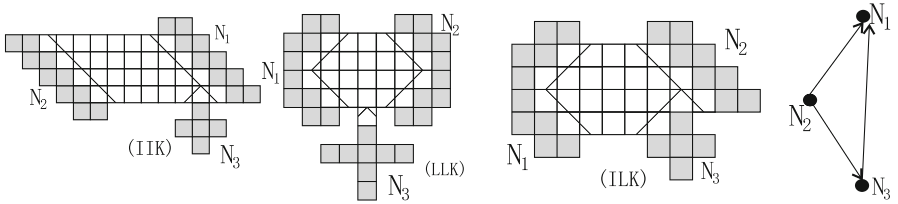


Fig. 15 Illustrations for polyomino graphs of $\mathcal{G}_3$ with label K and their NC-induced digraph

of H has at least 1 in-degree. So there is a directed cycle in H, contradicting Theorem 2.9. Note that this contradiction is caused by our assumption that there exists normal components in R. Therefore, the theorem holds. □

3 Classification and construction of $\mathcal{G}_n$

In this chapter, we deal with the construction of $\mathcal{G}_n$ with the aid of *special edge-cuts* [7]. Since the construction of $\mathcal{G}_n$ for $n \geq 4$ is more complex than we can imagine, in the following we consider mainly $\mathcal{G}_n$ for $n < 4$.

Let G be a polyomino graph with boundary C_0. Let E be a subset of the edge set $E(G)$ of G. E is said to be an *edge-cut* of G if $G - E$ is disconnected. In order to classify and construct $\mathcal{G}_2$ and $\mathcal{G}_3$, we need the following definitions.

Definition 3.27 (see [7]) A straight line segment P_1P_2 is called a *cut segment* if

1. each of P_1 and P_2 is the center of an edge on C;
2. P_1P_2 and each of the edges of G form an angle of $\pi/4$;
3. any point of P_1P_2 is either an interior or a boundary point of some square of G.

Definition 3.28 (see [7]) A broken line segment P_1QP_2 is called a *g-cut segment* if

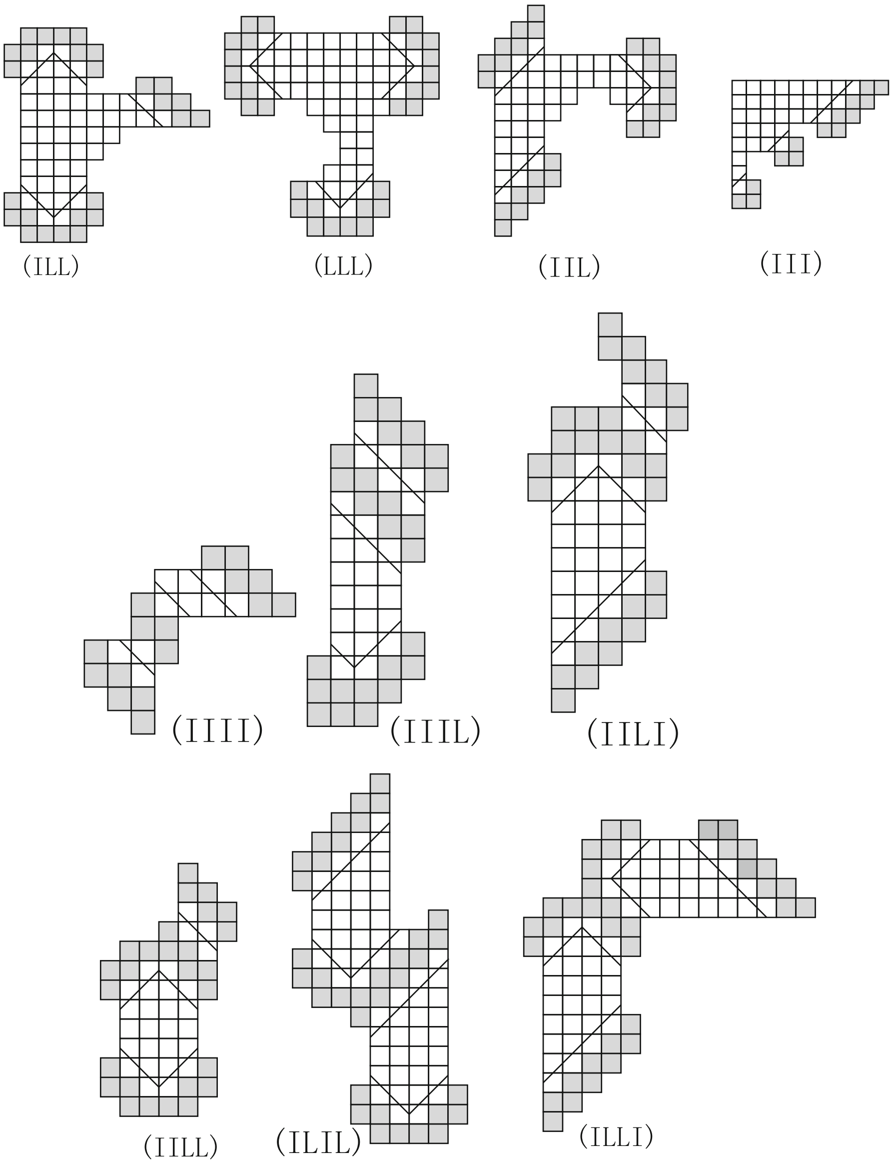


Fig. 16 Illustrations for polyomino graphs of $\mathcal{G}_3$ without label K

1. each of P_1 and P_2 is the center of an edge on C;
2. P_1Q is orthogonal to P_2Q; Q is the center of some edge e which is the bisector of the right angle $\angle P_1QP_2$;
3. any point of P_1QP_2 is either an interior or a boundary point of some square of G.

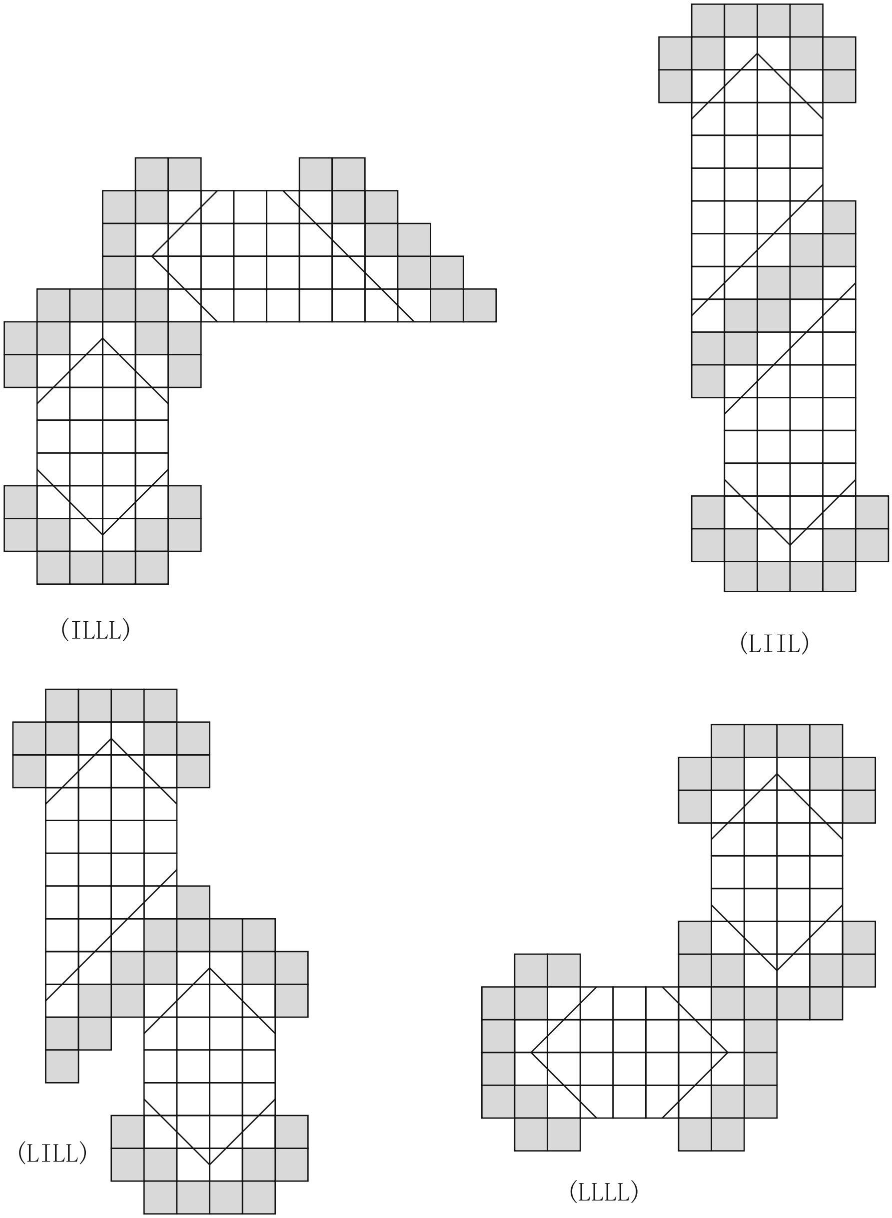


Fig. 16 continued

A *special cut segment* is either a cut segment or a g-cut segment. A *special edge-cut* is the set of edges of G intersected by a special cut segment. In Fig. 13 $P_{1a}P_{2a}$ is a cut segment, while $P_{1b}QP_{2b}$ is a g-cut segment. In the following with the aid of special edge-cuts, we discuss the classification and construction of essentially disconnected polyomino graphs of $\mathcal{G}_n (n = 2, 3)$.

It is know [8] that an edge of an essentially disconnected polyomino graph is a fixed single bond if and only if it is contained in some special edge-cut. Let N be a normal component of a polyomino graph G. Denote by $G - N$ the subgraph of G obtained by deleting all the vertices of N together with their incident edges. Clearly, all the edges each of which has one end vertex in N and the other end vertex in $G - N$ are fixed single bonds of G and form an edge cut of G, denoted by $(N, G - N)$.

Definition 3.29 Let N be a normal component of a polyomino graph $G \in \mathcal{G}_n$. An edge cut E incident with N is labeled by $I(L)$ if the edges of E correspond to a special cut segment (g-cut segment) of G. If the edges of E correspond to neither a special cut segment nor a special g-cut segment, E is labeled by K.

In the following chapter, we classify and construct polyomino graphs of $\mathcal{G}_2, \mathcal{G}_3$ according to the labels of edge cuts.

By the results in chapter 2, one can check that if N is of type B or W, edge cuts incident with N can only be labeled by I or L. Then we classify and construct polyomino graphs of $\mathcal{G}_2, \mathcal{G}_3$ and their NC-induced digraphs.

1. *For polyomino graphs of* $\mathcal{G}_2$*:*
   By the results in chapter 2, each member of $\mathcal{G}_2$ has exactly two normal components N_1 and N_2, one being of type W and the other being of type B. So polyomino graphs of $\mathcal{G}_2$ can be classified into three types : II, LL and IL, where XY means that the edge cut adjacent to N_1 is labeled by X, while the edge cut adjacent to N_2 is labeled by Y; $X \in \{I, L\}$, $Y \in \{I, L\}$. It is easy to check that all the polyomino graphs of $\mathcal{G}_2$ have the same NC-induced digraph (Fig. 14).
2. *For polyomino graphs of* $\mathcal{G}_3$*:*
   For polyomino graphs of $\mathcal{G}_3$ with label K, they have only three types: IIK, LLK, ILK (Fig. 15).

   For polyomino graphs of $\mathcal{G}_3$ without label K, they have fourteen types as follows: ILL, IIL, III, LLL; $IIII$, $IIIL$, $IILI$, $IILL$, $ILIL$, $ILLI$, $ILLL$, $LIIL$, $LILL$ and $LLLL$. Note that for the first four types (ILL, IIL, III, LLL), each normal component corresponds to only one special edge cut; while for other types, one of the three normal components corresponds to two special edge cuts (Fig. 16).

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